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Strong Instability of Solitary-Wave Solutions to a Kadomtsev–Petviashvili Equation in Three Dimensions

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This paper is concerned with strong instability of solitary-wave solutions of a generalized Kadomtsev–Petviashvili equation in the three-dimensional case

$$(u_t + u_{xxx} + u^p u_x)_x = u_{yy} + u_{zz} \qquad (x, y, z) \in R^3, \qquad t \geqslant 0,$$

1. INTRODUCTION

This paper is a continuation of a previous work [BoLiu] where we proved a nonlinear instability result with $1 \leqslant p < \frac{4}{3}$ for solitary waves of the generalized Kadomtsev–Petviashvili (KP) equation in the three-dimensional case (KP-3D),

$$(u_t + u_{xxx} + u^p u_x)_x = u_{yy} + u_{zz} \qquad (x, y, z) \in \mathbf{R}^3, \qquad t \geqslant 0,$$

or

$$\begin{aligned} u_t + u_{xxx} + u^p u_x &= v_y + w_z \\ v_x &= u_y \qquad (x, y, z) \in \mathbf{R}^3, \qquad t \geqslant 0, \\ w_x &= u_z \end{aligned} \tag{1.1}$$

where x is the direction of propagation and (y, z) are transverse variables.

KP equations are universal models for the propagation of weakly nonlinear dispersive long waves on the surface of a fluid, which is essentially one-directional with weak transverse effects [KaPe, PeYa]. To some

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extent, the KP equation (1.1) can be viewed as a three-dimensional analog of the generalized Korteweg–de Vries (GKdV) equation (see [KePoVe])

$$u_t + u_{xxx} + u^p u_x = 0.$$

The existence and stability of solitary waves is one of the most important mathematical questions of a nonlinear dispersive equation. There are many rigorous results that have recently appeared concerning the problems of local existence for the KP equation (for example, [Bou, FoSu, IsMeSt, Sa, Uk, Zh, To]). For KP-2D type equations, stability and instability of solitary waves are already established in [LiuWa, BouSa3, WaBaSe]. On the other hand, for KP-3D, by the conditions of negative energy and the power $p \geqslant 2$, one can show the solution of KP equation (1.1) blows up in finite time by using a virial identity [TuFa, Sa]. Recently, nonlinear instability of solitary waves were studied in [BoLiu] by using a variational method developed by Bona, Souganidis, and Strauss [BoSoSt, SoSt]. It is shown in [BoLiu] that solitary-wave solutions of KP equation (1.1) are nonlinearly unstable if $1 \leqslant p < 4/3$.

In this paper we are concerned with blow-up and strong instability of solitary waves of KP equation (1.1) for $p < 2$. It is shown that the solution of KP equation (1.1) blows up in finite time if $1 \leqslant p < 4/3$. Moreover, it can be proved that the solution blows up in finite time provided the initial data are close to an unstable solitary wave, that is, strong instability of solitary wave. The proofs are based on the idea in [Liu] for the KP-2D case by using a virial identity together with a construction of invariant sets for the flow which allow an optimal use of the virial identity.

Our motivation is based on the following fact: using the anisotropic Sobolev embedding [BeIlNi, p. 323], it is easy to see

$$\int_{\mathbf{R}^3} |u|^{p+2}\, dx\, dy\, dz \leqslant c\, \|u\|_{L^2}^{\frac{4-3p}{2}}\, \|\partial_x u\|_{L^2}^{\frac{3p}{2}}\, \|v\|_{L^2}^{\frac{p}{2}}\, \|w\|_{L^2}^{\frac{p}{2}} \tag{1.2}$$

which is valid only for $0 \leqslant p \leqslant 4/3$. It follows from (1.2) that $\int_{\mathbf{R}^3} ((u_x)^2 + v^2 + w^2)\, dx\, dy\, dz$ can be dominated by the conserved momentum $V(u) = \frac{1}{2}\int_{\mathbf{R}^3} u^2\, dx\, dy\, dz$ and the Hamiltonian

$$E(u) = \int_{\mathbf{R}^3} \left(\frac{1}{2} u_x^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) dx\, dy\, dz$$

if and only if $p < \frac{4}{5}$. So it is suggested that it is possible for the solutions to blow up even for $p \geqslant 4/5$. On the other hand, it is known in [BoSa1] that the existence of solitary waves has been established for $1 \leqslant p < 4/3$ in the space Y which is defined in Section 2 (Definition 2.1). Moreover, it was shown recently in [BoLiu] that such a solitary wave is nonlinearly unstable in Y for $1 \leqslant p < 4/3$. Therefore, it is possible to show that the solitary wave is strongly unstable for such a restriction of p.

In fact, we are able to prove in this paper that the solutions blow up in finite time in some sense ($\|\partial_y u(\cdot, t)\|_{L^2}^2 + \|\partial_z u(\cdot, t)\|_{L^2}^2 \to \infty$ in finite time), due to the weak transverse dispersion along the y and z-axes.

Remark. It is noted that the power $p \geqslant 1$ of the nonlinearity in KP-3D for the existence of solitary waves could be extended to $p > 4/5$ without any new assumptions. Therefore, one can obtain the results of strong instability of solitary waves for $4/5 < p < 4/3$.

In general, solutions of (1.1) satisfy the following conserved functionals.

$$E(u) = \int_{\mathbf{R}^3} \left(\frac{1}{2} u_x^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2 - \frac{1}{(p+1)(p+2)} u^{p+2} \right) dx\, dy\, dz \qquad \text{(Energy)}$$

$$V(u) = \frac{1}{2} \int_{\mathbf{R}^3} u^2 \, dx\, dy\, dz \qquad \text{(Momentum)}$$

$$\Phi_1(u) = \int_{\mathbf{R}^3} u \, dx\, dy\, dz,$$

$$\Phi_2(u) = \int_{\mathbf{R}^3} v \, dx\, dy\, dz, \qquad \text{and} \qquad \Phi_3(u) = \int_{\mathbf{R}^3} w \, dx\, dy\, dz.$$

We shall employ the following notations. $|\cdot|_p$ and $\|\cdot\|_s$ will denote the norms in $L^p(\mathbf{R}^3)$ and Sobolev space $H^s(\mathbf{R}^3)$, respectively. Let Y be the closure of $\partial_x(C_0^\infty(\mathbf{R}^3))$ with the norm

$$\|u\|_Y = \|\partial_x \psi\|_Y = (\|\nabla \psi\|_{L^2}^2 + \|\partial_x^2 \psi\|_{L^2}^2)^{\frac{1}{2}}$$

for $u \in Y$ and $u = \partial_x \psi$, where $\psi \in L^q_{loc}(\mathbf{R}^3)$, $\forall 2 \leqslant q < \infty$. We also have $v = \partial_y \psi \in L^2$ by a choice of $\psi \in L^q_{loc}$. Let

$$X_s = \{u \in H^s(\mathbf{R}^3);\ (\xi_1^{-1} \hat{u})^\vee \in H^s(\mathbf{R}^3)\}$$

with the norm $\|u\|_{X_s} = \|u\|_s + \|(\xi_1^{-1}\hat{u})^\vee\|_s$ and

$$\dot{H}^{-k} = \{u \in S'(\mathbf{R}^3),\ \xi_1^{-k} \hat{u} \in L^2(\mathbf{R}^3)\}$$

equipped with the norm $\|u\|_{-k,x} = |\xi_1^{-k}\hat{u}|_2$, where " $\wedge$ " is the Fourier transform and " $\vee$ " is the Fourier inverse transform.

Throughout this paper, we only consider the case when $p = n_1/n_2$ where n_1 is any even integer and n_2 any odd integer so that $\int_{\mathbf{R}^3} u^{p+2}\,dx\,dy = |u|_{p+2}^{p+2}$.

The plan of this paper is as follows. In Section 2, we study the properties of the solitary-wave solutions φ_c. In particular, we consider the associated minimization problem and employ a refined Fatou lemma due to Lieb and Brézis [BeLi1] to obtain the suitable minimizer, which is also known as *ground state*. In Section 3, some invariant sets for the flow of the KP equation are constructed. Then we are able to establish the blow up and strong instability results for $1 \leqslant p < \frac{4}{3}$ in Section 4.

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2. SOLITARY-WAVE SOLUTIONS

In this section, we study some properties of the solitary-wave solution φ_c to the KP equation (1.1). By the definition in [BoSa1], a traveling-wave solution of (1.1) means a form $u(t, x, y) = \varphi_c(x - ct, y, z)$ with $u \to 0$ as $x^2 + y^2 + z^2 \to \infty$, and φ_c is a ground state of the equation

$$\begin{aligned} -c\,\partial_x \varphi_c + \partial_x^3 \varphi_c - \partial_y \psi_c - \partial_z \phi_c + \varphi_c^p\,\partial_x \varphi_c &= 0, \\ \partial_x \psi_c &= \partial_y \varphi_c \\ \partial_x \phi_c &= \partial_z \varphi_c \qquad \varphi_c \in Y, \qquad \varphi_c \neq 0. \end{aligned} \tag{2.1}$$

Let $L_c(u) = E(u) + cV(u)$. We define the ground states of (2.1) as follows.

Definition 2.1. Let Γ_c be the set of the solutions of (2.1), namely,

$$\Gamma_c = \{\phi \in Y \mid L_c'(\phi) = 0,\ \phi \neq 0\},$$

and let G_c be the set of the ground states of (2.1), that is

$$G_c = \{\varphi \in \Gamma_c \mid L_c(\varphi) \leqslant L_c(\phi),\ \forall \phi \in \Gamma_c\}.$$

Remark. The existence of ground states for (2.1) was proved in [BoSa1]. It is shown that G_c is not empty for any $c > 0$ and $1 \leqslant p < 4/3$. Moreover, $\varphi_c \in H^\infty$ for $\varphi_c \in G_c$. But the uniqueness of the ground state is still open.

Local existence to the KP equation (1.1) has been studied in [Sa].

LEMMA 2.2. *Let $u_0 \in X_s$, $s \geqslant 3$, and $\partial^2_{yy} u_0, \partial^2_{zz} u_0 \in \dot{H}^{-2}_x$. Then there exists $T > 0$ such that Eq.* (1.1) *has a unique solution $u(t)$ with $u(0) = u_0$ satisfying*

$$u \in C([0, T); H^s(\mathbf{R}^3)) \cap C^1([0, T); H^{s-3}(\mathbf{R}^3))$$

and

$$v, w \in C([0, T); H^{s-1}(\mathbf{R}^3)).$$

Moreover, the energy $E(u)$ and the momentum $V(u)$ are well defined and independent of the time t.

Remark. By Lemma 3.2 in [Mo], assumptions of the initial data u_0 in Lemma 2.2 can be relaxed to $u_0 \in X_s$, $s > 2$ without any extra conditions.

DEFINITION 2.3. We say that the set $S \subset X$ is *X-stable*, if for any $\varepsilon > 0$, there exists $\delta > 0$ with $\inf_{w \in S} \|u_0 - w\|_X < \delta$ for any $u_0 \in X \cap X_s$ with $s > 2$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ can be extended to a global solution in $C([0, \infty); X \cap X_s)$ and satisfies

$$\sup_{0 \leqslant t < \infty} \inf_{w \in S} \|u(t) - w\|_X < \varepsilon.$$

Otherwise, S is called *X-unstable*. Also we say that the solitary wave φ_c is unstable if $O_c = \{\tau_{s,r,q} u \mid s, r, q \in \mathbf{R}, \tau_{s,r,q} u(x, y) = u(x+s, y+r, z+q)\}$ is *O_c-unstable*. Furthermore, we say that φ_c is strongly unstable if for any $\varepsilon > 0$, there exists $u_0 \in X$ such that the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time.

Remark. The set of ground states G_c with $c > 0$ for (1.1) is *Y-unstable* provided $1 \leqslant p < \frac{4}{3}$ is proved in [BoLiu].

To show blow up and strong instability of solutions of (1.1), we need a series of lemmas.

The first lemma is called a refined Fatou lemma due to Brézis and Lieb [BrLi1].

LEMMA 2.4. *Let $\{f_j\}$ be a bounded sequence in $L^r(\mathbf{R}^3)$ for $0 < r < \infty$. If $f_j \to f$ a.e. in $\mathbf{R}^3$, then*

$$|f_j|^r_r - |f_j - f|^r_r - |f|^r_r \to 0.$$

When $r = 2$, the assumption that $f_j \to f$ a.e. is not necessary.

The following lemma is given in [BoSa1, Lemma 3.3].

LEMMA 2.5. *Let u_n be a bounded sequence in Y, and let $R>0$. Then there is a subsequence u_{n_k} which converges strongly to some u in $L^2(B_R)$, where B_R is a ball with radius R in $\mathbf{R}^3$.*

LEMMA 2.6. *Let $u \in Y$ such that $\|u\|_Y \leqslant C$ and $\mu(|u|>\varepsilon) \geqslant \delta > 0$. Then there exists a shift $\tau_{s,r,q}u(x, y) = u(x+s, y+r, z+q)$ such that for some constant $\delta_0 = \delta_0(c, \delta, \varepsilon) > 0$,*

$$\mu\left(B \cap \left(|\tau_{s,r,q}u| > \frac{\varepsilon}{2}\right)\right) > \delta_0, \tag{2.3}$$

where B is the unit ball in $\mathbf{R}^3$.

Proof. We omit the proof, since it is the same as the proof of Lemma 4 in [LiuWa] with the space Y.

LEMMA 2.7. *Suppose $1 \leqslant p < 4/3$ and $c>0$. Let φ_c be a ground state. Then*

$$\text{(a)} \qquad 0 = K(\varphi_c) = \inf\left\{K(u) \mid u \in Y, \int_{\mathbf{R}^3} (\partial_x u)^2 = \int_{\mathbf{R}^3} (\partial_x \varphi_c)^2\right\}$$

$$\text{(b)} \quad I_c(\varphi_c) = \inf\left\{I_c(u) \mid u \in Y, \int_{\mathbf{R}^3} u^{p+2} = \int_{\mathbf{R}^3} \varphi_c^{p+2}\right\},$$

where

$$K(u) = \frac{1}{2}\int_{\mathbf{R}^3} (cu^2+v^2+w^2) + \frac{1}{6}\int_{\mathbf{R}^3} u_x^2 - \frac{1}{(p+1)(p+2)}\int_{\mathbf{R}^3} u^{p+2},$$

and $I_c(u) = \int_{\mathbf{R}^3} (cu^2+v^2+w^2+(\partial_x u)^2)\, dx\, dy\, dz$.

Proof. See Lemma 2.1 in [BoSa2].

LEMMA 2.8. *Let $1 \leqslant p < 4/3$ and $\varphi_c \in G_c$. Then*

$$\text{(a)} \quad L_c(\varphi_c) = \inf\left\{L_c(u) \mid u \in Y, u \neq 0, \int_{\mathbf{R}^3} u_x^2 = \int_{\mathbf{R}^3} (\partial_x \varphi_c)^2\right\} \tag{2.4}$$

$$\text{(b)} \quad L_c(\varphi_c) = \inf\left\{L_c(u) \mid u \in Y, u \neq 0, \int_{\mathbf{R}^3} u^{p+2} = \int_{\mathbf{R}^3} \varphi_c^{p+2}\right\}. \tag{2.5}$$

Proof. (a) Since $K = L_c(u) - \frac{1}{3}\int_{\mathbf{R}^3} u_x^2$, it follows from Lemma 2.7 that

$$\begin{aligned}
&\inf\left\{L_c(u)\,|\,u\in Y,\, u\neq 0,\, \int_{\mathbf{R}^3} u_x^2 = \int_{\mathbf{R}^3} (\partial_x\varphi_c)^2\right\}\\
&\quad = \inf\left\{K(u)+\frac{1}{3}\int_{\mathbf{R}^3}(\partial_x\varphi_c)^2\,|\,u\in Y,\, u\neq 0,\, \int_{\mathbf{R}^3} u_x^2 = \int_{\mathbf{R}^3} (\partial_x\varphi_c)^2\right\}\\
&\quad = \inf\left\{K(u)\,|\,u\in Y,\, u\neq 0,\, \int_{\mathbf{R}^3} u_x^2 = \int_{\mathbf{R}^3} (\partial_x\varphi_c)^2\right\}+\frac{1}{3}\int_{\mathbf{R}^3}(\partial_x\varphi_c)^2\\
&\quad = \frac{1}{3}\int_{\mathbf{R}^3}(\partial_x\varphi_c)^2 = L_c(\varphi_c).
\end{aligned}$$

(b)
$$\begin{aligned}
&\inf\left\{L_c(u)\,|\,u\in Y,\, u\neq 0,\, \int_{\mathbf{R}^3} u^{p+2} = \int_{\mathbf{R}^3} \varphi_c^{p+2}\right\}\\
&\quad = \frac{1}{2}\inf\left\{\int_{\mathbf{R}^3}(cu^2+v^2+w^2+u_x^2)\,|\,u\in Y,\, u\neq 0,\, \int_{\mathbf{R}^3} u^{p+2} = \int_{\mathbf{R}^3} \varphi_c^{p+2}\right\}\\
&\quad\quad -\frac{1}{(p+1)(p+2)}\int_{\mathbf{R}^3}\varphi_c^{p+2} = \frac{1}{2}I_c(\varphi_c) - \frac{1}{(p+1)(p+2)}\int_{\mathbf{R}^3}\varphi_c^{p+2}\\
&\quad = L_c(\varphi_c).
\end{aligned}$$

The following lemma is crucial to obtain the invariant set for the flow of the KP equation.

LEMMA 2.9. *Assume $1 \leqslant p < 4/3$. Let $\sigma = \inf\{L_c(u)\,|\,u\in M\}$, where $M = \{u\in Y\,|\,u\neq 0,\, Q(u)=0\}$ and*

$$Q(u) = \int_{\mathbf{R}^3}\left(v^2+w^2+(\partial_x u)^2 - \frac{5p}{2(p+1)(p+2)}u^{p+2}\right)dx\,dy\,dz. \tag{2.6}$$

Then φ_c is a ground state for (2.1) *if and only if $\varphi_c\in M$ and $L_c(\varphi_c)=\sigma$.*

Proof. Let φ_c be a ground state and define $\phi_c^\lambda(x,y,z) = \lambda^{\frac{5}{2}}\varphi_c(\lambda x, \lambda^2 y, \lambda^2 z)$, $\lambda > 0$. Then

$$Q(\varphi_c) = \frac{d}{d\lambda}L_c(\phi_c^\lambda)|_{\lambda=1} = \left\langle L_c'(\varphi_c), \left.\frac{d\phi_c^\lambda}{d\lambda}\right|_{\lambda=1}\right\rangle = 0. \tag{2.7}$$

This implies that $\varphi_c \in M$ for any $\varphi_c \in G_c$. Now define

$$(2.8)\qquad L_c^1(u) = L_c(u) - \frac{2}{5p} Q(u) = \frac{c}{2}\int_{\mathbf{R}^3} u^2 + \frac{5p-4}{10p}\int_{\mathbf{R}^3} (v^2+w^2+u_x^2).$$

We claim $\sigma = m$, where

$$(2.9)\qquad m = \inf\{L_c^1(u) \mid u \in Y, u \neq 0, Q(u) \leqslant 0\}.$$

Suppose $Q(u) < 0$. Since

$$(2.10)\quad Q(\lambda u) = \lambda^2 \int_{\mathbf{R}^3} (v^2+w^2+u_x^2) - \frac{5p}{2(p+1)(p+2)} \lambda^{p+2} \int_{\mathbf{R}^3} u^{p+2} > 0$$

for some sufficiently small $\lambda > 0$, there exists $\lambda_0 \in (0, 1)$ such that $Q(\lambda_0 u) = 0$. Hence we have

$$\begin{aligned}\sigma \leqslant L_c(\lambda_0 u) &= \lambda_0^2 \left(\frac{c}{2}\int_{\mathbf{R}^3} u^2 + \frac{5p-4}{10p}\int_{\mathbf{R}^3} (v^2+w^2+u_x^2)\right)\\ &< \frac{c}{2}\int_{\mathbf{R}^3} u^2 + \frac{5p-4}{10p}\int_{\mathbf{R}^3} (v^2+w^2+u_x^2) = L_c^1(u).\end{aligned}$$

Consequently, $m = \sigma$. To show that $\sigma = L_c(\varphi_c)$ for some $\varphi_c \in G_c$, it suffices to show that $m = L^1(\varphi_c)$ for some $\varphi_c \in G_c$. In fact, we have $L_c^1(u) > 0$, $1 \leqslant p < 4/3$. Hence, there exists a minimizing sequence $\{u_j\}$ of (2.9) satisfying that $\{u_j\}$ is bounded in Y, $L_c^1(u_j) \to m$, and $Q(u_j) \leqslant 0$. By the anisotropic Sobolev embedding [BeIlNi, p. 323],

$$|u|_{p+2} \leqslant c\,\|u\|_Y \qquad \text{for} \quad 0 \leqslant p \leqslant 4/3.$$

It turns out that $\{u_j\}$ is bounded in L^{p+2} for $0 \leqslant p < 4/3$. We then have some subsequences, still denoted by $\{u_j\}$ and $u_0 \in Y \cap L^{p+2}$ such that u_j weakly converges to u_0 in Y and in L^{p+2} for $0 \leqslant p < 4/3$. It follows from Lemma 2.7 that $u_j \to u_0$ a.e. in $\mathbf{R}^3$. Now we are able to show $L_c^1(u_0) = m$ and $Q(u_0) = 0$. Toward this end, we split the proof into the following five steps.

STEP 1. $\inf_j |u_j|_{p+2}^{p+2} > 0$.

Proof of Step 1. Suppose there exists a subsequence of $\{u_j\}$ such that $|u_j|_{p+2}^{p+2} \to 0$. Then from $Q(u_j) \leqslant 0$ we obtain

$$|v_j|_2^2 + |w_j|_2^2 + |\partial_x u_j|_2^2 \leqslant \frac{5p}{2(p+1)(p+2)} \int_{\mathbf{R}^3} u_j^{p+2} \to 0.$$

On the other hand, by the facts that $Q(u_j) \leqslant 0$ and the anisotropic Sobolev embedding [BeLiNi]

$$|u|_{p+2}^{p+2} \leqslant C\, |u|_2^{\frac{4-3p}{2}}\, |v|_2^{\frac{p}{2}}\, |w|_2^{\frac{p}{2}}\, |\partial_x u|_2^{p} \qquad \text{for} \quad 0 \leqslant p \leqslant 4/3. \tag{2.11}$$

It follows that

$$\begin{aligned} |v_j|_2^2 + |w_j|_2^2 + |\partial_x u_j|_2^2 &\leqslant C\, |u_j|_2^{\frac{4-3p}{2}} \left(|v_j|_2^2 + |w_j|_2^2 + |\partial_x u_j|_2^2\right)^{\frac{5p}{4}} \\ &\leqslant C\left(|v_j|_2^2 + |w_j|_2^2 + |\partial_x u_j|_2^2\right)^{\frac{5p}{4}}, \end{aligned}$$

since $|u_j|_2$ is bounded. Therefore,

$$\left(|v_j|_2^2 + |w_j|_2^2 + |\partial_x u_j|_2^2\right)\left(1 - C\left(|v_j|_2^2 + |\partial_x u_j|_2^2\right)^{\frac{5p-4}{4}}\right) \leqslant 0.$$

This implies that

$$1 \leqslant C\left(|v_j|_2^2 + |w_j|_2^2 + |\partial_x u_j|_2^2\right)^{\frac{5p-4}{4}}. \tag{2.12}$$

But it contradicts $|v_j|_2^2 + |\partial_x u_j|_2^2 \to 0$, because $p \geqslant 1 > \frac{4}{5}$. Consequently, $\inf_j |u_j|_{p+2}^{p+2} > 0$.

STEP 2. $u_0 \neq 0$ *a.e.*

Proof of Step 2. Let $\inf_j |u_j|_{p+2}^{p+2} = \alpha > 0$. We estimate

$$\begin{aligned} \alpha \leqslant |u_j|_{p+2}^{p+2} &= \int_{|u_j| \geqslant \frac{1}{\varepsilon}} |u_j|^{p+2} + \int_{|u_j| \leqslant \varepsilon} |u_j|^{p+2} + \int_{\varepsilon < |u_j| < \frac{1}{\varepsilon}} |u_j|^{p+2} \\ &\leqslant \int_{|u_j| \geqslant \frac{1}{\varepsilon}} \frac{|u_j|^{p+2+\gamma}}{|u_j|^{\gamma}} + \varepsilon^p \int_{|u_j| \leqslant \varepsilon} |u_j|^2 + \left(\frac{1}{\varepsilon}\right)^{p+2} \mu(|u_j| > \varepsilon) \\ &\leqslant \varepsilon^{\gamma} \int_{|u_j| \geqslant \frac{1}{\varepsilon}} |u_j|^{p+2+\gamma} + \varepsilon^p \int_{|u_j| \leqslant \varepsilon} |u_j|^2 + C_\varepsilon \mu(|u_j| > \varepsilon), \end{aligned} \tag{2.13}$$

where $0<\gamma<4/3-p$. Since $p+\gamma<4/3$, it follows from the anisotropic Sobolev embedding [BeIlni]

$$\int_{|u_j|\geqslant\frac{1}{\varepsilon}}|u_j|^{2+p+\gamma}\leqslant\int_{\mathbf{R}^2}|u_j|^{2+p+\gamma}\leqslant C(|u_j|_2^2+|v_j|_2^2+|\partial_x u_j|_2^2)^{\frac{2+p+\gamma}{2}}\leqslant C_1$$

and

$$\int_{|u_j|\geqslant\frac{1}{\varepsilon}}|u_j|^2\leqslant\int_{\mathbf{R}^3}|u_j|^2\leqslant C_2,$$

where C, C_1, and C_2 are some constants. Choosing ε sufficiently small, we obtain

$$\mu(|u_j|>\varepsilon)\geqslant\frac{\alpha-\varepsilon^{\gamma}C_1-\varepsilon^{p}C_2}{C_\varepsilon}=\delta>0.$$

It follows from Lemma 2.6 that $\mu(B\cap(|u_0|>\frac{\varepsilon}{2}))>\delta_0$ for the unit ball B because $u_j\to u_0$ a.e. in $\mathbf{R}^3$. This implies that $u_0\neq 0$ a.e. in $\mathbf{R}^3$.

STEP 3. $m=L_c^1(u_0)$.

Proof of Step 3. By Lemma 2.4, we deduce that

$$Q(u_j)-Q(u_j-u_0)-Q(u_0)\to 0, \tag{2.14}$$

and

$$L_c^1(u_j)-L_c^1(u_j-u_0)-L_c^1(u_0)\to 0. \tag{2.15}$$

Now suppose that $Q(u_0)>0$. Then from the fact that $Q(u_j)\leqslant 0$ we obtain that $Q(u_j-u_0)\leqslant 0$ as $j\to\infty$. By the definition of m, it follows that $L_c^1(u_j-u_0)\geqslant m$. Since $L_c^1(u_j)\to m$, it follows from (2.13) that $L_c^1(u_0)\leqslant 0$. That is

$$\frac{c}{2}\int_{\mathbf{R}^3}u_0^2+\frac{5p-4}{10p}\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2)\leqslant 0. \tag{2.16}$$

This contradicts $u_0\neq 0$, a.e. Consequently, $Q(u_0)\leqslant 0$. Therefore

$$m\leqslant L_c^1(u_0)\leqslant\lim_{j\to\infty}\inf L_c^1(u_j)=m$$

and $m=L_c^1(u_0)$.

STEP 4. $Q(u_0)=0$.

Proof of Step 4. Suppose that $Q(u_0)<0$. We try to get a contradiction. Toward this end, we choose a small $\lambda>0$ and find that

$$Q(\lambda u_0)=\lambda^2\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2)-\frac{5p}{2(p+1)(p+2)}\lambda^{p+2}\int_{\mathbf{R}^3}u_0^{p+2}>0.$$

It follows from the continuity that there exists $\lambda_0\in(0,1)$ such that $Q(\lambda_0 u_0)=0$. Therefore by the definition of m, it yields a contradiction in the following

$$\begin{aligned}m\leqslant L_c^1(\lambda_0 u_0)&=\lambda_0^2\left(\frac{c}{2}\int_{\mathbf{R}^3}u_0^2+\frac{5p-4}{10p}\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2\right)\\&<\frac{c}{2}\int_{\mathbf{R}^3}u_0^2+\frac{5p-4}{10p}\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2)\\&=L_c^1(u_0)=m.\end{aligned}$$

This proves that $Q(u_0)=0$.

STEP 5. $u_0=\varphi_c\in G_c$.

Proof of Step 5. It follows from Steps 3 and 4 that

$$m=\sigma=\inf\{L_c(u)\,|\,u\neq 0,\ Q(u)=0\}=L_c(u_0). \tag{2.17}$$

Hence, there exists a constant $\lambda\in\mathbf{R}$ such that

$$L_c'(u_0)+\lambda Q'(u_0)=0.$$

It remains to show that $\lambda=0$ so that $L_c'(u_0)=0$. In fact, let $w^\eta(x,y,z)=\eta^{5/2}u_0(\eta x,\eta^2 y,\eta^2 z)$. It is easy to see that

$$\begin{aligned}0&=\left\langle (L_c'+\lambda Q')(u_0),\left.\frac{\partial w^\eta}{\partial\eta}\right|_{\eta=1}\right\rangle=\partial_\eta(L_c+\lambda Q)(w^\eta)|_{\eta=1}\\&=Q(u_0)+\lambda\left(2\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2)-\frac{25p^2}{4(p+1)(p+2)}\int_{\mathbf{R}^3}u_0^{p+2}\right).\end{aligned}\tag{2.18}$$

If $\lambda \neq 0$, then

$$2\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2)-\frac{25p^2}{4(p+1)(p+2)}\int_{\mathbf{R}^3} u_0^{p+2}=0.$$

Since $Q(u_0)=0$, it follows that

$$\begin{aligned}0&=2\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2)-\frac{5p}{2}\int_{\mathbf{R}^3}(v_0^2+w_0^2+(\partial_x u_0)^2)\\&=\frac{4-5p}{2}\int_{\mathbf{R}^3} v_0^2+w_0^2+(\partial_x u_0)^2<0\end{aligned}$$

because $p\geqslant 1$. Hence $\lambda=0$ and $L'(u_0)=0$. On the other hand, suppose $u\in Y$ satisfies $L_c'(u)=0$. We have

$$Q(u)=\partial_\eta L_c(u^\eta)|_{\eta=1}=\left\langle L_c'(u), \left.\frac{du^\eta}{d\eta}\right|_{\eta=1}\right\rangle=0,$$

where $u^\eta=\eta^{5/2}u(\eta x, \eta^2 y, \eta^2 z)$. It follows from (2.15) that $L_c(u_0)\leqslant L_c(u)$. This implies that $u_0=\varphi_c\in G_c$. The proof of Lemma 2.9 is complete.

3. INVARIANT SETS

In this section we construct some invariants for the flow of the KP equation (1.1). Using those invariant properties of the solution and a virial identity which has been shown in [TuFa], we are able to show the blow-up result . Toward this end, we begin to define the invariants in the following:

$$K_1^c=\{u\in Y\,|\,u\neq 0,\, L_c(u)<L_c(\varphi_c),\, Q(u)\geqslant 0\},$$

$$K_2^c=\{u\in Y\,|\,u\neq 0,\, L_c(u)<L_c(\varphi_c),\, Q(u)<0\},$$

$$R_1^c=\left\{u\in Y\,|\,u\neq 0,\, L_c(u)<L_c(\varphi_c),\, \int_{\mathbf{R}^3}u_x^2\leqslant\int_{\mathbf{R}^3}(\partial_x\varphi_c)^2\right\},$$

$$R_2^c=\left\{u\in Y\,|\,u\neq 0,\, L_c(u)<L_c(\varphi_c),\, \int_{\mathbf{R}^3}u_x^2>\int_{\mathbf{R}^3}(\partial_x\varphi_c)^2\right\},$$

$$J_1^c=\left\{u\in Y\,|\,u\neq 0,\, L_c(u)<L_c(\varphi_c),\, \int_{\mathbf{R}^3}u^{p+2}\geqslant\int_{\mathbf{R}^3}\varphi_c^{p+2}\right\}\qquad\text{and}$$

$$J_2^c=\left\{u\in Y\,|\,u\neq 0,\, L_c(u)<L_c(\varphi_c),\, \int_{\mathbf{R}^3}u^{p+2}<\int_{\mathbf{R}^3}\varphi_c^{p+2}\right\}.$$

The following lemma is key to obtain the blow-up result.

LEMMA 3.1 (Invariant sets). *Suppose* $1 \leqslant p < 4/3$ *and* $c > 0$. *Let* u_0 *be the initial data such that the corresponding solution* $u(t)$ *of KP equation* (1.1) *is in* $C([0,T); Y)$ *for some* $T > 0$ *and satisfies* $E(u(t)) = E(u_0)$ *and* $V(u(t)) = V(u_0)$ *for* $0 \leqslant t < T$. *Then*

(a) $u_0 \in K_i^c$ *implies that* $u(t) \in K_i^c$, $\forall 0 \leqslant t < T$,

(b) $u_0 \in R_i^c$ *implies that* $u(t) \in R_i^c$, $\forall 0 \leqslant t < T$, *and*

(c) $u_0 \in J_i^c$ *implies that* $u(t) \in J_i^c$, $\forall 0 \leqslant t < T$ *where* $i = 1, 2$. *Moreover if* $u_0 \in K_2^c$, *then* $Q(u(t)) < -\frac{5p}{2}(L_c(\varphi_c) - L_c(u_0))$ *for* $0 \leqslant t < T$.

Proof. Here we only consider the invariance of K_2^c, since for K_1^c the proof is similar and the proof of invariance of R_1^c and J_i^c are also similar because of Lemma 2.8.

Let $u_0 \in K_2^c$. Since $E(u(t)) = E(u_0)$ and $V(u(t)) = V(u_0)$, we have

$$L_c(u(t)) = E(u(t)) + cV(u(t)) = L_c(u_0) < L_c(\varphi_c). \tag{3.1}$$

Suppose $u(t_0) \notin K_2^c$ for some $t_0 \in (0, T)$, that is $Q(u(t_0)) \geqslant 0$. By $Q(u(0)) = Q(u_0) < 0$ and the continuity of $Q(u(t))$ with respect to t, there exists $t_1 \in (0, t_0]$ such that $Q(u(t_1)) = 0$. Therefore applying Lemma 2.9 yields a contradiction

$$L_c(\varphi_c) > L_c(u(t_1)) \geqslant \inf\{L_c(u) \mid u \neq 0,\ Q(u) = 0\} = L_c(\varphi_c).$$

This implies that $u(t) \in K_2^c$ for $0 \leqslant t < T$. To prove the final inequality, we use the definition of m and the fact

$$m = \inf\{L_c(u) \mid u \in Y, u \neq 0,\ Q(u) = 0\} = L_c(\varphi_c)$$

which is proved in Lemma 2.9.

Suppose $u_0 \in K_2^c$. Then we have $u(t) \in K_2^c$; i.e., $L_c(u(t)) < L_c(\varphi_c)$ and $Q(u(t)) < 0$ for $t \geqslant 0$. Since

$$Q(\lambda u) = \lambda^2 \int_{\mathbf{R}^3} (v^2 + w^2 + u_x^2) - \frac{5p}{2(p+1)(p+2)} \lambda^{p+2} \int_{\mathbf{R}^3} u^{p+2} > 0$$

for some sufficiently small $\lambda > 0$, there exists $\lambda_0 \in (0, 1)$ such that $Q(\lambda_0 u) = 0$ and

$$L_c(\varphi_c) \leqslant L_c(\lambda_0 u) = \lambda_0^2 \left(\frac{c}{2} \int_{\mathbf{R}^3} u^2 + \frac{5p-4}{10p} \int_{\mathbf{R}^3} (v^2 + w^2 + u_x^2) \right)$$
$$< \frac{c}{2} \int_{\mathbf{R}^3} u^2 + \frac{5p-4}{10p} \int_{\mathbf{R}^3} (v^2 + w^2 + u_x^2) = L_c^1(u(t)).$$

Therefore, $Q(u(t)) < -\frac{5p}{2}(L_c(\varphi_c) - L_c(u_0))$.

4. BLOW-UP AND STRONG INSTABILITY

It was shown in [Sa] that the solution of KP-3D (1.1) blows up in finite time provided the energy is negative and $p \geqslant 2$. Using the virial identity with a construction of the invariant sets, we are able to extend this blow-up result to allow the energy E even positive and also $1 \leqslant p < 4/3$.

THEOREM 4.1 (Improved blow-up). *Let $1 \leqslant p < 4/3$ and $c > 0$. Assume*

(i) *$u_0 \in X_s$, $s > 2$ and $yu_0 \in L^2$, and*

(ii) *$u_0 \in K_2^c \cap R_2^c \cap J_2^c$.*

Let u be the solution of the KP equation (1.1) in $C([0, T); H^s)$ with $u(0) = u_0$, conserved energy E, and momentum V. Then there exists a blow-up time $T_0 < \infty$ such that

$$\lim_{t \to T_0^-} (|u_y|_2^2 + |u_z|_2^2) = \infty. \tag{4.1}$$

Proof. Define $I(t) = \int_{\mathbf{R}^3} (y^2 + z^2)\, u^2\, dx\, dy\, dz$, where u is the solution of the KP equation (1.1). Using the virial identity in [TuFa, Sa] yields

$$\begin{aligned} \frac{d^2 I(t)}{dt^2} &= 8pE(u(t)) + (2-p) \int_{\mathbf{R}^3} (v^2 + w^2)\, dx\, dy\, dz \\ &\quad - 4p \int_{\mathbf{R}^3} u_x^2\, dx\, dy\, dz \\ &= 8 \left(Q(u(t)) - \int_{\mathbf{R}^3} u_x^2 + \frac{3p}{2(p+1)(p+2)} \int_{\mathbf{R}^3} u^{p+2} \right). \end{aligned} \tag{4.2}$$

It follows from Lemma 3.1 that

$$\frac{d^2I(t)}{dt^2} < 8\left(-\frac{5p}{2}(L_c(\varphi_c)-L_c(u_0))-\int_{\mathbf{R}^3}(\partial_x\varphi_c)^2 + \frac{3p}{2(p+1)(p+2)}\int_{\mathbf{R}^3}\varphi_c^{p+2}\right) = -20p\varepsilon_0, \tag{4.3}$$

where $\varepsilon_0 = L_c(\varphi_c)-L_c(u_0)$. In the last step of the above proof, we used the fact

$$\int_{\mathbf{R}^3}(\partial_x\varphi_c)^2 = \frac{3p}{2(p+1)(p+2)}\int_{\mathbf{R}^3}\varphi_c^{p+2}. \tag{4.4}$$

It follows that $\lim_{t\to T_0} I(t) = 0$ for some $T_0 < \infty$, and the blow-up result can be deduced from the conserved momentum $V(u)$ and the classical inequality

$$|u|_2^2 \leqslant 2\,|(y^2+z^2)^{\frac{1}{2}}u|_2\,(|\partial_y u|_2^2+|\partial_z u|_2^2).$$

The proof of Theorem 4.1 is complete.

Remark. It is easy to see that the energy

$$E(\varphi_c) = \frac{5p-4}{6p}\,|\partial_x\varphi_c|_2^2 > 0.$$

So it is possible to choose the initial data $u_0 \in K_2^c \cap R_2^c \cap J_2^c$ which is close to the solitary wave φ_c such that the energy $E(u_0) > 0$.

A strong instability of solitary-wave solutions φ_c can be obtained in the following.

THEOREM 4.2. *Let $1 \leqslant p < 4/3$. Suppose φ_c is the solitary-wave solution of the KP equation* (1.1) *with $c > 0$. For any $\delta > 0$, there is an initial data $u_0 \in X_s$, $s > 2$ with $\|u_0 - \varphi_c\|_Y < \delta$, such that the solution u of* (1.1) *with $u(0) = u_0$ blows up in finite time. More precisely,*

$$\lim_{t\to T^-}(|\partial_y u(t)|_2^2 + |\partial_z u(t)|_2^2) = \infty.$$

Proof of Theorem 4.2. We define $u_0(x, y, z) = \lambda\varphi_c(\mu x, \xi y, \xi z)$, where $\xi^2 = (1-\varepsilon)\,\mu^4$ with a sufficiently small $\varepsilon > 0$. By Theorem 4.1, it suffices to show that u_0 is close to the solitary wave φ_c for small ε and $u_0 \in K_2^c \cap R_2^c \cap J_2^c$, that is, φ_c satisfies the following conditions

(1) $L_c(u_0) < L_c(\varphi_c)$,
(2) $Q(u_0) < 0$,
(3) $|\partial_x u_0|_2^2 > |\varphi_c|_2^2$, and
(4) $\int_{\mathbf{R}^3} u_0^{p+2} < \int_{\mathbf{R}^3} \varphi_c^{p+2}$.

It can be verified from the facts of $L_c(\varphi_c) = \frac{1}{3}\,|\partial_x \varphi_c|_2^2$, $c\,|\varphi_c|_2^2 = \frac{4-3p}{3p}\,|\partial_x \varphi_c|_2^2$, $|\partial_x^{-1}\,\partial_y \varphi_c|_2^2 = |\partial_x^{-1}\,\partial_z \varphi_c|_2^2 = \frac{1}{3}\,|\partial_x \varphi_c|_2^2$, and $\frac{3p}{2(p+1)(p+2)} \int_{\mathbf{R}^3} \varphi_c^{p+2} = |\partial_x \varphi_c|_2^2$, that condition (1) is equivalent to

(c_1) $\frac{4-3p}{6p}\mu^{-1}\xi^{-2} + \frac{1}{2}\mu\xi^{-2} + \frac{1}{3}\mu^{-3} - \frac{2}{3p}\mu^{-1}\xi^{-2}\lambda^p < \frac{1}{3}\lambda^{-2}$, condition (2) is equivalent to
(c_2) $\xi^{-2}\mu^2 + \frac{2}{3}\mu^{-2} - \frac{5}{3}\lambda^p\xi^{-2} < 0$, condition (3) is equivalent to
(c_3) $\lambda^2\mu\xi^{-2} > 1$, and condition (4) is equivalent to
(c_4) $\lambda^{p+2}\mu^{-1}\xi^{-2} < 1$.

To verify condition (c_1) it suffices to show the following condition

$$(c_1') \qquad \frac{4-3p}{6p}\,\mu^{-1}\xi^{-2} + \frac{5p-4}{10p}\left(\mu\xi^{-2} + \frac{2}{3}\,\mu^{-3}\right) < \frac{1}{3}\,\lambda^{-2}$$

due to condition (c_2).

It is shown from a simple computation that all of the conditions c_1', c_2, c_3, and c_4 are satisfied if

$$\left(\frac{5(4-3p)}{5(4-3p) + 2(5p-4)\,\varepsilon - 10p(1-(1+\varepsilon)^{-2\eta})}\right)^{\frac{1}{2}} < \mu < (1-\varepsilon)^{\frac{p}{4-3p}}\left(\frac{5}{5-2\varepsilon}\right)^{\frac{2}{4-3p}}$$

and $\lambda = (1+\varepsilon)^{\eta}\,(1-\varepsilon)^{\frac{1}{2}}\,\mu^{\frac{3}{2}}$ with sufficiently small $\varepsilon > 0$ and $\eta > 0$. It is noted that

$$\left(\frac{5(4-3p)}{5(4-3p) + 2(5p-4)\,\varepsilon - 10p(1-(1+\varepsilon)^{-2\eta})}\right)^{\frac{1}{2}} > (1-\varepsilon)^{\frac{p}{4-3p}}\,(1+\varepsilon)^{\frac{2(p+2)\,\eta}{4-3p}}$$

with $\eta \to 0$, $\varepsilon \to 0$. On the other hand, it is easy to verify that $\lambda \to 1$, $\mu \to 1$, and $u_0 \to \varphi_c$ in Y, as $\varepsilon \to 0$. The proof of Theorem 4.2 is complete.

Remark. In [Liu], we proved the solitary wave φ_c is strongly unstable if $2 < p < 4$ for the KP$-$2D equation. In fact, by choosing suitable initial data $u_0 = \lambda\varphi_c(\mu x, \xi y)$ as we did in Theorem 4.2, one can improve the result of strong instability to $4/3 \leqslant p < 4$.

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